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LETTER TO THE EDITOR

Biased ultradiffusion

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Abstract. A model of ultradiffusion on a one-dimensional bifurcating hierarchical structure with a hierarchical set of biasing terms is introduced. By an exact renormalization decimation procedure, a new non-universal time scaling exponent for the autocorrelation function and a new crossover for the dynamical phase transition, from ordinary to anomalous diffusion, are obtained. We also find a new transition for the autocorrelation function from power law to exponential decay which depends on the distribution of the biasing terms. Hierarchically biased diffusion on multifurcating hierarchical structures are also discussed briefly.

Since Huberman and Kerzberg (1985) proposed the one-dimensional ultradiffusion model for relaxation in hierarchical structures, various aspects on hierarchical structures have been considered, including electronic properties (Ceccatto *et al* 1987), vibrational spectrum (Keirstead *et al* 1988), multifractal nature (Havlin and Matan 1988, Kahng and Redner 1989, Lin and Tao 1990), etc., and several generalized versions have been introduced (Ceccatto and Riera 1986, Ceccatto and Huberman 1988, Giacometti *et al* 1988, Zheng *et al* 1989). It is believed that hierarchical, or ultrametric, organization should play a very important role in different physical contexts ranging from molecular diffusion (Austin *et al* 1975) to spin glasses (Sompolinsky 1981) and computing architectures (Huberman and Hogg 1984).

The earliest 1D ultradiffusion model proposed by Huberman and Kerzberg (1985) for relaxation in hierarchical structures consists of random walks of a particle in a one-dimensional chain with a regular uniformly bifurcating hierarchical array of barriers, and it has been treated within a renormalization group scheme (Maritan and Stella 1986). The anomalous exponent x for the long-time behaviour of the autocorrelation function $P_0(t) \sim t^{-x/2}$ was exactly given by

$$x = 2\ln/\ln[2(2W_1^* + W_0)/W_1^*] \tag{1}$$

where W_1^* characterizes a whole line of fixed points to which the initial barrier hierarchy $\{W_n\}$ is attracted. For the particular hierarchy $W_j = R^j$ $(j = 0, 1, \ldots)$, (1) becomes

$$x = \begin{cases} 2\ln 2/(\ln 2 - \ln R) & 0 < R < 1/2 \\ 1 & 1/2 < R < 1 \end{cases} \tag{2}$$

and one obtains a crossover $R_c = 1/2$ for the dynamical phase transition from ordinary to anomalous diffusion.

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As a generalization of the above model, Ceccatto and Riera (1986) studied the 1D ultradiffusion version with a uniformly biasing term η. For $\min\{W_n\} < W_0$, they found that the behaviour (1) of the autocorrelation function is not modified by the bias. When $\min\{W_n\} = W_0$, however, they found a new transition in the relaxational behaviour of the autocorrelation function, from power law to exponential decay.

Recently, Keirstead *et al* (1987) and Zheng *et al* (1989) considered ultradiffusion on general K-furcating hierarchical structures and concluded that the anomalous diffusion exponent x and the phase transition point from ordinary to anomalous diffusion in the case of particular hierarchy depend on the furcating number of the hierarchy. In the case of biased ultradiffusion however, we will show that the transition from power law to exponential decay of the autocorrelation function does not depend on the multifurcation number of the structure, but on the distribution of the biasing terms.

In this letter, we present and analyse, by an exact renormalization method, a general biased diffusion model which has a hierarchical set of biasing terms $\{\eta_i\}$ $(i = 1, 2, \ldots)$ as shown in figure 1. Some interesting results are obtained and the corresponding expressions for the corresponding multifurcating hierarchical systems are also suggested.

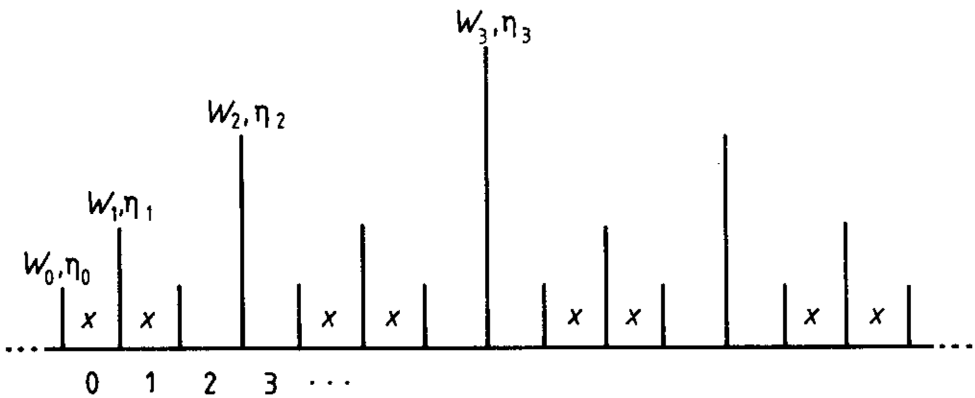

Figure 1. Bifurcating hierarchical barrier structure with a hierarchical set of biasing terms $\{\eta_n\}$. The cells with crosses are decimated in the renormalization group procedure.

If we call $P_n(t)$ the probability of finding the particle at time t at cell n, its Laplace transform $\tilde{P}_n(\omega)$ satisfies the following equation:

$$\begin{aligned}\omega\tilde{P}_n &= \delta_{n,0} + (W_{n-1,n} + \eta_{n-1,n})\tilde{P}_{n-1} + (W_{n+1,n} - \eta_{n+1,n})\tilde{P}_{n+1} \\ &\qquad - (W_{n,n-1} - \eta_{n,n-1})\tilde{P}_n - (W_{n,n+1} + \eta_{n,n+1})\tilde{P}_n \\ &= \delta_{n,0} + W_{n-1,n}(\tilde{P}_{n-1} - \tilde{P}_n) + W_{n,n+1}(\tilde{P}_{n+1} - \tilde{P}_n) \\ &\qquad + \eta_{n-1,n}(\tilde{P}_{n-1} + \tilde{P}_n) - \eta_{n+1,n}(\tilde{P}_{n+1} + \tilde{P}_n) \end{aligned} \tag{3}$$

where the rates $W_{n,n+1}$ and the biasing terms $\eta_{n,n+1}$ are assigned hierarchically as pictured in figure 1. They take the appropriate values W_j and η_j respectively, associated with the corresponding barrier. $\delta_{n,0}$ means that the particle is supposed to be in cell 0 at the initial time. One can easily recognize that the model seems reasonable only for $\eta_i \leqslant W_i$ $(i = 1, 2, \ldots)$.

Performing an exact renormalization group decimation procedure which eliminates the cells marked by crosses in figure 1, we obtained a new system of the same form as (3) with the following recursion relations:

$$W_j' = W_{j+1}/\Delta \qquad \eta_j' = \eta_{j+1}/\Delta \qquad j = 1, 2, \ldots$$
$$\eta_0' = \eta_0/\Delta \qquad \tilde{P}_n' = \Delta \tilde{P}_n \qquad \omega' = \Omega\omega \tag{4}$$

where

$$\Delta = \frac{W_0^2(W_1 + \eta_0 - \eta_1) + \eta_0^2(W_1 + \eta_1) + 2W_0\eta_0\eta_1 - \eta_0^3}{W_0(W_0^2 + 2W_0W_1 + 2\eta_0\eta_1 - \eta_0^2)} \tag{5}$$

and

$$\Omega = \frac{2W_0[W_0^2 + 2W_0W_1 + \eta_0\eta_1 - \eta_0(W_0 + W_1)]}{W_0^2(W_1 + \eta_0 - \eta_1) + \eta_0^2(W_1 + \eta_1) + 2W_0\eta_0\eta_1 - \eta_0^3}. \tag{6}$$

It is worth pointing out that in deriving the above recursion relations we have used the condition $W_0' = W_0$ to fix the unit of time and considered only the $\omega \to 0$ limit to obtain the leading dynamical scaling behaviour of the autocorrelation function $P_0(t)$.

The recursion $\eta_0' = \eta_0\Delta$ has two interesting fixed points $\eta_0^* = 0$ and $\eta_0^* = W_0$. For the fixed point $\eta_0^* = 0$, one easily obtains

$$\Delta^* = \frac{W_1^* - \eta_1^*}{W_0 + 2W_1^*} \qquad \Omega^* = \frac{2(W_0 + 2W_1^*)}{W_1^* - \eta_1^*} \tag{7}$$

and the non-universal time scaling exponent x for the autocorrelation function $P_0(t) \sim t^{-x/2}$ can be found to be

$$x = 2\ln/\ln[2(2W_1^* + W_0)/(W_1^* - \eta_1^*)] \tag{8}$$

where W_1^* and η_1^* characterize the lines of fixed points to which the initial $\{W_n\}$ and $\{\eta_n\}$ are attracted respectively. The exponent varies continuously between $x = 0$ (trapping) and $x = 1$ (normal diffusion) with $W_1^* - \eta_1^*$ ranging from 0 to $+\infty$ (notice that W_1^* varies from 0 to $+\infty$, and η_1^* from 0 to W_1^*). For the particular sets of transition rates $W_j = R^j$ $(j = 1, 2, \ldots)$ and biasing terms $\eta_i = \rho W_i$ $(0 \leq \rho \leq 1, i = 1, 2, \ldots)$, one has

$$x = \begin{cases} 2\ln 2/(\ln 2 - \ln R) & 0 < R < (1-\rho)/2 \\ 1 & (1-\rho)/2 < R < 1. \end{cases} \tag{9}$$

Therefore the dynamical transition from ordinary to anomalous diffusion is at $R_c = (1-\rho)/2$ in the present case. Following Zheng *et al* (1989), it is reasonable to suggest the corresponding expressions of (8) and (9) as

$$x = K\ln K/\ln[K(KW_1^* + W_0)/(W_1^* - \eta_1^*)] \tag{10}$$

and

$$x = \begin{cases} K\ln K/\ln(K - \ln R) & 0 < R < (1-\rho)/K \\ 1 & (1-\rho)/K < R < 1 \end{cases} \tag{11}$$

respectively, for a corresponding general K-furcating hierarchical system. Equation (11) gives the crossover $R_c = (1-\rho)/K$.

Now we turn to the fixed point $\eta_0^* = W_0$, which will result in a new transition for the autocorrelation function, from power law to exponential decay. Denoting by $n = 0$

the left-hand cell marked with a cross in figure 1, the exponential decay for the autocorrelation function can be exactly found as

$$P_0(t) = (C_1\,e^{-\lambda_1 t} + C_2\,e^{-\lambda_2 t})/2 \tag{12}$$

where

$$\lambda_{1,2} = (W_0 + W_1) \pm (W_0^2 + W_1^2 - 2W_0\eta_1)^{1/2} \tag{13}$$

and

$$C_{1,2} = 1 \pm (\eta_1 - W_0)(W_0^2 + W_1^2 - 2W_0\eta_1)^{1/2}. \tag{14}$$

Comparing the present results with those obtained by Ceccatto and Riera (1986) and by Zheng *et al* (1989), one may conclude that this new transition from power law to simple exponential decay of $P_0(t)$ depends on the distribution of the biasing terms, but not on the furcating number of the hierarchy.

In summary, we have proposed a one-dimensional ultradiffusion model with a hierarchical set of transition rates and a hierarchical set of biasing terms. Through an exact renormalization method, a new anomalous diffusion exponent x for the autocorrelation function $P_0(t)$ and a new crossover R_c for the dynamical phase transition from ordinary to anomalous diffusion were obtained. We also found a new transition for the autocorrelation function $P_0(t)$, from power law to exponential decay, which depends on the distribution of the biasing terms. Besides, the corresponding results for multifurcating hierarchical systems have also been suggested.

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